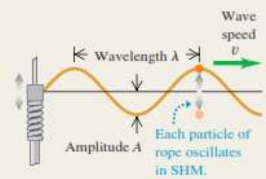
# Seminary 8

# MECHANICAL WAVES

**Waves and their properties:** A wave is any disturbance that propagates from one region to another. A mechanical wave travels within some material called the medium. The wave speed $v$ depends on the type of wave and the properties of the medium.

In a periodic wave, the motion of each point of the medium is periodic with frequency $f$ and period $T$. The wavelength $\lambda$ is the distance over which the wave pattern repeats, and the amplitude $A$ is the maximum displacement of a particle in the medium. The product of $\lambda$ and $f$ equals the wave speed. A sinusoidal wave is a special periodic wave in which each point moves in simple harmonic motion. (See Example 15.1.)

$$v = \lambda f \quad (15.1)$$




**Wave functions and wave dynamics:** The wave function $y(x, t)$ describes the displacements of individual particles in the medium. Equations (15.3), (15.4), and (15.7) give the wave equation for a sinusoidal wave traveling in the $+x$-direction. If the wave is moving in the $-x$-direction, the minus signs in the cosine functions are replaced by plus signs. (See Example 15.2.)

The wave function obeys a partial differential equation called the wave equation, Eq. (15.12).

The speed of transverse waves on a string depends on the tension $F$ and mass per unit length $\mu$. (See Example 15.3.)

$$y(x, t) = A\cos\left[\omega\left(\frac{x}{v} - t\right)\right] = A\cos 2\pi f\left(\frac{x}{v} - t\right) \quad (15.3)$$

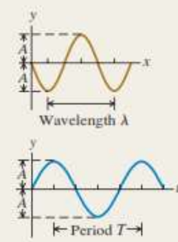
$$y(x, t) = A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \quad (15.4)$$

$$y(x, t) = A\cos(kx - \omega t) \quad (15.7)$$

where $k = 2\pi/\lambda$ and $\omega = 2\pi f = vk$

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x, t)}{\partial t^2} \quad (15.12)$$

$$v = \sqrt{\frac{F}{\mu}} \quad \text{(waves on a string)} \quad (15.13)$$



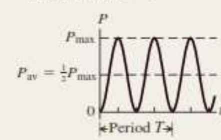

**Wave power:** Wave motion conveys energy from one region to another. For a sinusoidal mechanical wave, the average power $P_{av}$ is proportional to the square of the wave amplitude and the square of the frequency. For waves that spread out in three dimensions, the wave intensity $I$ is inversely proportional to the square of the distance from the source. (See Examples 15.4 and 15.5.)

$$P_{av} = \tfrac{1}{2}\sqrt{\mu F}\,\omega^2 A^2 \quad (15.25)$$

(average power, sinusoidal wave)

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \quad (15.26)$$

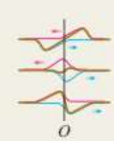
(inverse-square law for intensity)




**Wave superposition:** A wave reflects when it reaches a boundary of its medium. At any point where two or more waves overlap, the total displacement is the sum of the displacements of the individual waves (principle of superposition).

$$y(x, t) = y_1(x, t) + y_2(x, t) \quad (15.27)$$

(principle of superposition)




**Standing waves on a string:** When a sinusoidal wave is reflected from a fixed or free end of a stretched string, the incident and reflected waves combine to form a standing sinusoidal wave with nodes and antinodes. Adjacent nodes are spaced a distance $\lambda/2$ apart, as are adjacent antinodes. (See Example 15.6.)

When both ends of a string with length $L$ are held fixed, standing waves can occur only when $L$ is an integer multiple of $\lambda/2$. Each frequency with its associated vibration pattern is called a normal mode. (See Examples 15.7 and 15.8.)

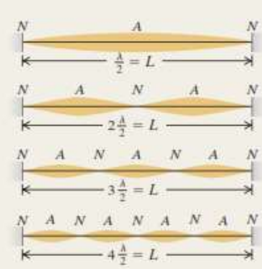
$$y(x, t) = (A_{SW}\sin kx)\sin\omega t \quad (15.28)$$

(standing wave on a string, fixed end at $x = 0$)

$$f_n = n\frac{v}{2L} = nf_1 \ (n = 1, 2, 3, \ldots) \quad (15.33)$$

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \quad (15.35)$$

(string fixed at both ends)

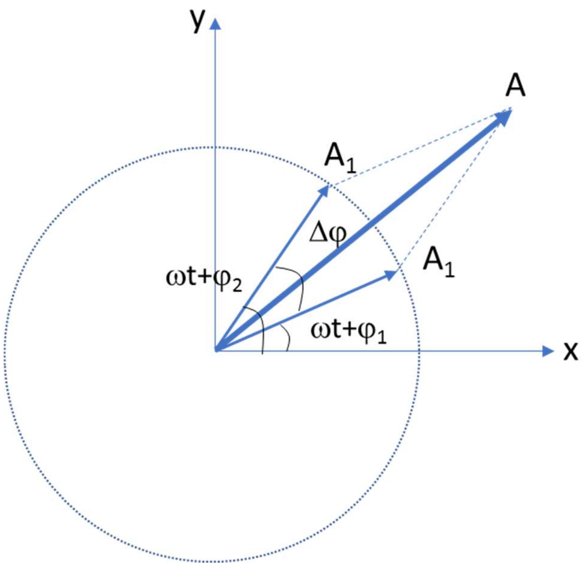



**Extracted from:**
**SEARS AND ZEMANSKY'S UNIVERSITY PHYSICS,** WITH MODERN PHYSICS 13TH EDITION.

## WAVE SUPERSOPSITION/STANDING WAVES

**1/ Phasors:**

Two sinusoidal waves of the same frequency travel in the same direction along a string. If $A_1$=3.0 cm, $A_2$= 4.0 cm, $\varphi_1$ = 0,and $\varphi_2$= $\pi/2$ rad, what is the amplitude of the resultant wave?




*Phasor representation*:

$$\begin{cases} y_1(t) = A\sin(\omega_1 t + \varphi_1); \\ y_2(t) = A\sin(\omega_2 t + \varphi_2) \end{cases}$$

$$y(t) = y_1(t) + y_2(t)$$

$$A^2 = A_1^{\ 2} + A_2^{\ 2} + 2A_1A_2\cos(\Delta\varphi)$$

$$\Delta\varphi = (\omega_2 - \omega_1)t + (\varphi_2 - \varphi_1)$$

**2/** A standing wave is produced by superposition of an incident and a reflected propagative waves described by the following equations:

$$y_1(x,t) = -A\cos(kx + \omega t) \quad \text{(incident wave traveling to the left)}$$
$$y_2(x,t) = A\cos(kx - \omega t) \quad \text{(reflected wave traveling to the right)}$$

(a) Using the principle of superposition and the trigonometric equation:

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

demonstrate that the standing wave pattern on a string fixed at x=0 can be described by the equation:

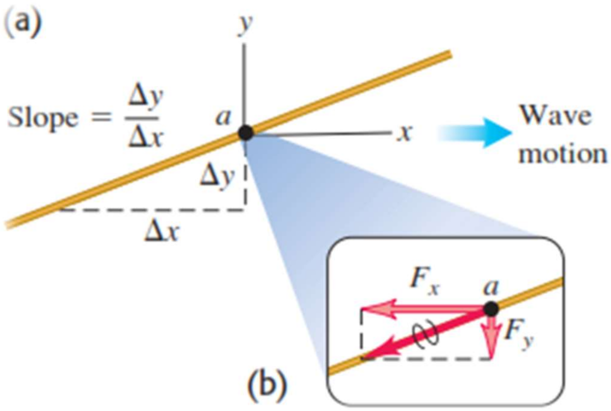
$$y(x,t) = (A_{SW}\sin kx)\sin\omega t$$ with $A_{SW} = 2A$ .

Graphically represent and discuss this result.

(b) Evaluating the power at a point in a string, as the product between the transverse force and the transverse velocity of that point:

$$P(x,t) = F_y(x,t)v_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial t}$$

and using the wave function of a standing wave, demonstrate that the average power carried by a standing wave is zero.



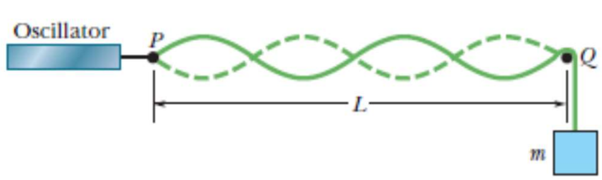

$$\frac{F_y}{F} = -slope = -\frac{\partial y(x,t)}{\partial x}$$

**3/** Adjacent antinodes of a standing wave on a string are 15.0 cm apart. A particle at an antinode oscillates in simple harmonic motion with amplitude 0.850 cm and period 0.0750 s. The string lies along the –axis and is fixed at x=0 (a) How far apart are the adjacent nodes? (b) What are the wavelength, amplitude, and speed of the two traveling waves that form this pattern? (c) Find the maximum and minimum transverse speeds of a point at an antinode. (d) What is the shortest distance along the string between a node and an antinode?

**4/** A piano tuner stretches a steel piano wire with a tension of 800 N. The steel wire is 0.400 m long and has a mass of 3.00 g. (a) What is the frequency of its fundamental mode of vibration? (b) What is the number of the highest harmonic that could be heard by a person who is capable of hearing frequencies up to 10000 Hz?

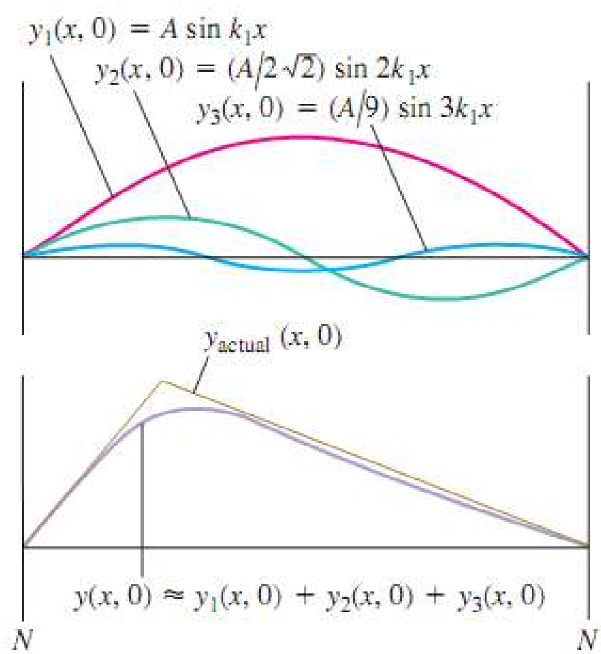
**5/** In the following series of resonant frequencies, one frequency (lower than 400 Hz) is missing: 150, 225, 300, 375 Hz. (a) What is the missing frequency? (b) What is the frequency of the seventh harmonic?

**6/** In Fig. below, a string, tied to a sinusoidal oscillator at $P$ and running over a support at $Q$, is stretched by a block of mass $m$. Separation $L$=1.20 m, linear density $\mu$= 1.6 g/m, and the oscillator frequency $f$ = 120 Hz. The amplitude of the motion at $P$ is small enough for that point to be considered a node. A node also exists at $Q$. (a) What mass $m$ allows the oscillator to set up the fourth harmonic on the string? (b) What standing wave mode, if any, can be set up if $m$ = 1.00 kg?



**Discussion on Complex Standing Waves: Fourier analysis**

If we could displace a string so that its shape is the same as one of the normal-mode patterns and then release it, it would vibrate with the frequency of that mode. Such a vibrating string would displace the surrounding air with the same frequency, producing a traveling sinusoidal sound wave that your ears would perceive as a pure tone. But when a string is struck (as in a piano) or plucked (as is done to guitar strings), the shape of the displaced string is *not* as simple. The fundamental as well as many overtones are present in the resulting vibration. This motion is therefore a combination or *superposition* of many normal modes. Several simple harmonic motions of different frequencies are present simultaneously, and the displacement of any point on the string is the sum (or superposition) of the displacements associated with the individual modes. The sound produced by the vibrating string is likewise a superposition of traveling sinusoidal sound waves, which you perceive as a rich, complex tone with the fundamental frequency f1.The standing wave on the string and the traveling sound wave in the air have similar **harmonic content** (the extent to which frequencies higher than the fundamental are present). The harmonic content depends on how the string is initially set into motion. If you pluck the strings of an acoustic guitar in the normal location over the sound hole, the sound that you hear has a different harmonic content than if you pluck the strings next to the fixed end on the guitar body.
It is possible to represent every possible motion of the string as some superposition of normal-mode motions. Finding this representation for a given vibration pattern is called *harmonic analysis.* The sum of sinusoidal functions that represents a complex wave is called a *Fourier series.* Figure below shows how a standing wave that is produced by plucking a guitar string of length *L* at a point from one end can be represented as a combination of sinusoidal functions.




*When a guitar string is plucked (pulled into a triangular shape) and released, a standing wave results. The standing wave is well represented (except at the sharp maximum point) by the sum of just three sinusoidal functions. Including additional sinusoidal functions further improves the representation.*

When plucking the string, it is removed by a distance *h* at position *d* from its equilibrium state. The shape of the string the moment it is plucked defines a function f (x) .


$$f(x) = \begin{cases} \frac{hx}{d}, & 0 \le x \le d \\ \frac{h(L-x)}{L-d}, & d < x \le L \end{cases}$$

**Fourier transform:**

Any *smooth* function f(x) has a unique representation

$$f(x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi}{L}x\right)$$

Where the coefficients are computed by:

$$A_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx$$

Integration by parts formula:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

Fourier coefficients:

$$A_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx = \frac{2}{L}\Bigg[\underbrace{\int_0^d \frac{hx}{d}\sin\left(\frac{k\pi x}{L}\right)dx}_{\text{Part 1}} + \underbrace{\int_d^L \frac{h(L-x)}{L-d}\sin\left(\frac{k\pi x}{L}\right)dx}_{\text{Part 2}}\Bigg]$$